Inner Product Spaces - II

(0) A couple of important things:

- Be careful with quantifiers. Most of you forgot them in Assignment 3. I did not cut marks this time, but this is a very VERY important issue.
- Words of encouragement: Check this out before the midterm http://www.math.vanderbilt.edu/~msapir/tt.html. I suggest you listen to this in a quiet place, preferably in the dark, at least three times a day.

(1) Let $k = \mathbb{C}$ or $\mathbb{R}$. Let $V = M_2(k)$ be the space of 2×2 matrices with entries from k. V has a natural inner product space structure give by $\langle A, B\rangle = tr(B^*A)$ for $A, B \in V$.

(a) (Unimportant question.) Why is this inner product "natural"? What is the reason behind defining it like this?

(b) We say that a matrix A is self-adjoint if $A^* = A$. Show that the set S of all self-adjoint 2×2 matrices form a subspace of V. Find the dimension of S. (You already did this exercise several times in the first half of this course, when $k = \mathbb{R}$.) **Important Note:** In tutorial, we quickly realized that if the underlying field is $\mathbb{C}$, S is not a subspace. Do this exercise over $\mathbb{R}$.

(c) Find an orthonormal basis of S.

(d) Let Q be the set of all matrices in V with the following property:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & d \\ a & c \end{bmatrix}$$

Show that Q is a subspace and find a basis for $Q^\perp$

(2) Label the following statements as true or false and justify your answer.

(a) Let V be a finite dimensional inner product space and W be a subspace of V. If $V = W \oplus U$ for a subspace U of W, then U is the orthogonal complement of W.

(b) Let V be a vector space and W_1, W_2, W_3 be subspaces of V. If $V = W_1 \oplus W_2$ and $V = W_1 \oplus W_3$, then we have $W_2 = W_3$.

(c) Let V be a finite dimensional vector space and U, W be subspaces of V. Let α, β be bases for U and W, respectively. If $|\alpha \cup \beta| = \dim V$, then $V = U \oplus W$.

(d) Let V be a finite dimensional vector space and U, W be subspaces of V. Let α, β be bases for U and W, respectively. If $|\alpha \cup \beta| = \dim V$ and $\alpha \cup \beta$ is linearly independent, then $V = U \oplus W$.

(e) Let V be a finite dimensional inner product space and $T : V \to V$ be a linear operator. If $\langle T(x), x\rangle = 0$ for all $x \in V$, then T is the zero transformation.

(f) Let V be a finite dimensional inner product space and $T : V \to V$ be a linear operator. If $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$, then T is invertible.

(g) Let V be a finite dimensional vector space and $T : V \to V$ be a linear operator. Then, $V = \ker T \oplus \mathrm{im} T$.

(h) Let V be a finite dimensional inner product space and $T : V \to V$ be a linear operator. Let β be a basis for V. If $[T]_\beta$ is a self-adjoint matrix, then T is a self-adjoint transformation.